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EXPRESSIONS FOR THE ELEMENTS OF A DETERMINANT IN TERMS OF THE MINORS OF A GIVEN ORDER. GENERALIZATION OF A THEOREM DUE TO STUDNICKA.

By L. E. DICKSON.

I. It is desired to express the n^2 elements a_{ij} of a determinant $D \equiv |a_{ij}|$ in terms of the $C^2_{n,m}$ minors of order m. Arrange these minors in a determinant $\triangle$ so that there shall appear in one row [column] all the $C_{n,m}$ minors built from the elements of the same m rows [columns] of D; then $\triangle$ is called the mth compound of D. It is well known that

$$(1) \qquad D^{C_{n-1,\ m-1}} = \triangle .$$

For the applications in view it suffices to assume that $\triangle \neq 0$, whence $D \neq 0$.

One symmetrical method of treating the problem consists in determining all the minors M_{m-1} of order $m-1$, then all the minors M_{n-2}, ..., and finally the a_{ij}. We first determine D from (1) by a root extraction.[*] By Studnicka's theorem, a very special case of formula (6) proved below, $M_{m-1}^{n-m}D$ can be expressed rationally in terms of the M_m, so that any minor M_{m-1} can be found by root extraction.

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[*] We can however find D by the extraction of a dth root, where d is the greatest common divisor of m and n, since D^d equals a sum of products of the M_m. For instance, if $d=1$, this follows from Laplace's expansion of D in the form $\Sigma_{\neq} M'_m M''_m \ldots\ldots M_m^{(n/m)}$, where M'_m is formed from the first m rows of D, M''_m from the second m rows, etc. Again, if $d=2$ take $n=6$, $m=4$ for brevity of writing; then

Such a method may properly be rejected since it introduces an array of radicals of various orders. It is in fact possible to express the a_{ij} in terms of the minors M_m so that the *only* irrationality entering is the mth root of a certain rational function of the M_m. It is evident *a priori* that the a_{ij} are at least m valued functions of the M_m. For, if the a'_{ij} give one set of solutions and if ϵ is an mth root of unity, then the products $\epsilon a'_{ij}=a''_{ij}$ give another set of solutions, since $|\ a''_{ij}\ |_m=\epsilon^m\ |\ a'_{ij}\ |_m$. We shall henceforth demand solutions a_{ij} which are exactly† m-valued functions of the M_m.

2. I give first a very elementary method of solution. For general m I rely upon the argument now given for the simplest case $m=n-1$. Let A_{ij} be the co-factor (minor of order $n-1$ with prefixed sign) of a_{ij} in D. Then $\triangle=|\ A_{ij}\ |=D^{n-1}$, so that D is the mth root of a rational function of the M_m. It is well known that the determinant of order $n-1$, obtained from $\triangle$ by deleting the ith row and jth column, equals $D^{n-2}a_{ij}$, apart from sign. Hence the a_{ij} are rational functions of the M_m and of the mth root of a certain rational function of the M_m.

3. By way of illustration, consider the case $m=2$ (n arbitrary), so that the minors M_2 of order 2 are given. From formulae like

$$(2)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^2 = \begin{vmatrix} A_{33} & A_{32} & A_{31} \\ A_{23} & A_{22} & A_{21} \\ A_{13} & A_{12} & A_{11} \end{vmatrix},$$

where A_{ij} is the minor of a_{ij} in $|\ a_{ij}\ |$, we can test whether or not a given M_3 vanishes. Since the M_3 built from the first three rows of D do not all vanish, we assume that, for example, the determinant in the left member of (2) is not

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$$D^2=\begin{vmatrix} D & O \\ O & D \end{vmatrix}=\left|\begin{array}{cccccc} 11 & 12 & \ldots\ldots 16 & 0 & 0 & \ldots\ldots 0 \\ 21 & 22 & \ldots\ldots 26 & 0 & 0 & \ldots\ldots 0 \\ 31 & 32 & \ldots\ldots 36 & 0 & 0 & \ldots\ldots 0 \\ 41 & 42 & \ldots\ldots 46 & 0 & 0 & \ldots\ldots 0 \\ \hline 51 & 52 & \ldots\ldots 56 & 51 & 52 & \ldots\ldots 56 \\ 61 & 62 & \ldots\ldots 66 & 61 & 62 & \ldots\ldots 66 \\ 11 & 12 & \ldots\ldots 16 & 11 & 12 & \ldots\ldots 16 \\ 21 & 22 & \ldots\ldots 26 & 21 & 22 & \ldots\ldots 26 \\ \hline 0 & 0 & \ldots\ldots 0 & 31 & 32 & \ldots\ldots 36 \\ 0 & 0 & \ldots\ldots 0 & 41 & 42 & \ldots\ldots 46 \\ 0 & 0 & \ldots\ldots 0 & 51 & 52 & \ldots\ldots 56 \\ 0 & 0 & \ldots\ldots 0 & 61 & 62 & \ldots\ldots 66 \end{array}\right|,$$

as shown by adding the 1st row to the 7th, 2d to 8th, 11th to 5th, and 12th to 6th. By Laplace's expansion, the last is expressed in terms of the M_4. It is readily shown that D^d is the lowest power of D which is rational in the M_m.

† A direct proof that the solutions are expressible as m-valued functions is given in *Linear Groups*, p. 146, §154. I did not consider there the present problem of exhibiting the solutions.

zero. Then by §2, the nine elements a_{11}, ..., a_{33} can be readily expressed rationally in terms of the M_2 and the square root of the second determinant (2).

Not all the minors A_{33}, A_{32}, A_{31} vanish, say $A_{32} \neq 0$. Having the minors $a_{11}a_{2i}-a_{21}a_{1i}$ and $a_{13}a_{2i}-a_{23}a_{1i}$, we get a_{2i} and a_{1i} since the determinant of their coefficients is $-A_{32}$. In this way we can determine a_{1i}, a_{2i}, a_{3i}, a_{i1}, a_{i2}, a_{i3} ($i=4$, ..., n). Of the elements previously found let $a_{13} \neq 0$, for example. We may now find a_{rs} ($r>3$, $s>3$) from $a_{13}a_{rs}-a_{1s}a_{r3}$.

4. As a further illustration, consider the case $m=3$. Of the determinants of order 4 built from the first four rows of D, suppose that

$$(3) \qquad \begin{vmatrix} 12 & 13 & 14 & 15 \\ \cdots & \cdots & \cdots & \cdots \\ 42 & 43 & 44 & 45 \end{vmatrix} \neq 0.$$

By §2, the sixteen elements of (3) are readily expressed as rational functions of the minors M_3 and the cube root of a rational function of the M_3. We next determine a_{1r}, a_{2r}, a_{3r} ($r=1$, 6, 7, ..., n). Consider the six M_3 :

$$a_{3r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} - a_{2r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{3i} & a_{3j} \end{vmatrix} + a_{1r}\begin{vmatrix} a_{2i} & a_{2j} \\ a_{3i} & a_{3j} \end{vmatrix} \qquad (i, j=2, 3, 4, 5;\ i<j).$$

From three of these M_3 we can find a_{3r}, $-a_{2r}$, a_{1r} if[*] the determinant of their coefficients is $\neq 0$. For $i, j=2$, 3; 2, 4; 3, 4, this determinant is

$$\begin{vmatrix} \begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 32 & 33 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 32 & 33 \end{vmatrix} \\ \begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} & \begin{vmatrix} 12 & 14 \\ 32 & 34 \end{vmatrix} & \begin{vmatrix} 22 & 24 \\ 32 & 34 \end{vmatrix} \\ \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 22 & 23 & 24 \\ 32 & 33 & 34 \end{vmatrix}^2 .$$

Indeed, by interchanging rows and columns in the first, we have a special case of (1). For $i, j=2$, 3; 2, 5; 3, 5, the determinant equals

$$\begin{vmatrix} 12 & 13 & 15 \\ 22 & 23 & 25 \\ 32 & 33 & 35 \end{vmatrix}^2 ,$$

etc. But not all minors of order 3 built from the first three rows of (3) vanish. Thus a_{1r}, a_{2r}, a_{3r} may be considered to be found. Of the minors from the first two rows of (3), let $a_{13}a_{24}-a_{14}a_{23} \neq 0$. We may thus get a_{4r} from

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[*] This determinant vanishes if $i, j=2$, 3; 2, 4; 2, 5.

$$\begin{vmatrix} 1r & 13 & 14 \\ 2r & 23 & 24 \\ 4r & 43 & 44 \end{vmatrix}.$$

Similarly, we can find every a_{s2}, a_{s3}, a_{s4}, a_{s5}. Finally, from

$$\begin{vmatrix} 1l & 13 & 14 \\ 2l & 23 & 24 \\ kl & k3 & k4 \end{vmatrix}$$

we determine a_{kl} (k any $\neq 1, 2, 3, 4$; l any $\neq 2, 3, 4, 5$).

5. The following method of solution for the important case $m=2$ offers greater symmetry than that of §3. We show that, if $t, i_1, i_2, \dots\dots, i_{n-2}$ are any positive integers $\overline{\gtrless} n$, $a_{ti_1}, a_{ti_2}, \dots\dots, a_{tin-2}D$ equals a certain determinant of order $n-1$ whose elements are all minors of degree 2 of $D \equiv | a_{ij} |$. Let j_1 be any integer $\neq i_1$, of the set $1, \dots\dots, n$; let j_2 be any one $\neq i_2$ or j_1; j_3 any $\neq i_3, j_1, j_2$; $\dots\dots$; j_{n-2} any $\neq i_{n-2}, j_1, j_2, \dots\dots, j_{n-1}$; j_{n-1} and j_n the remaining two. Set $i_{n-1} = j_n$. Call C_i the ith column in D. Subtract $a_{tj_1}C_{i_1}$ from $a_{ti_1}C_{j_1}$, $a_{tj_2}C_{i_2}$ from $a_{ti_2}C_{j_2}$, $\dots\dots$, $a_{tjn-1}C\,in-1$ from $a_{tin-1}C\,jn-1$. Hence $a_{ti_1}\dots\dots a_{tin-1}D$ equals a determinant the elements of whose j_nth column are the same as in D, while the element lying in the j_sth column ($s<n$) and rth row is

$$a_{ti_s}\,a_{rj_s} - a_{tj_s}\,a_{ri_s} \equiv T_{rj_s}.$$

Hence in the tth row all the elements are zero except that in the j_nth column. Thus the new determinant is a_{tj_n} times its co-factor. Hence

$$(4)\quad a_{ti_1}\dots a_{ti_{n-2}}D = (-1)^{t+j_n} \,|\, T_{rj_s} \,|\quad (s=1, \dots, n-1;\ r=1, \dots, t-1, t+1, \dots, n),$$

the sequence of columns being that of the order of magnitude of $j_1, \dots\dots, j_{n-1}$. By interchanging the rows and columns of D we obtain similarly

$$(5)\qquad a_{i_1t}\dots a_{i_{n-2}}D = \text{rational functions of minors of degree 2.}$$

Giving fixed values to $i_2, \dots, i_{n-2}$, and varying i_1, we conclude that the ratio of any two elements a of the same row (or column) is a known rational function of the minors of degree 2. Thus if $a_{t\sigma} \neq 0$, $a_{tl}=R_l=a_{t\sigma}$, $a_{k\sigma}=R_k'a_{t\sigma}$, for every l, k. To prove that $a^2{}_{t\sigma}$ is a rational function of the minors of degree 2, consider any non-vanishing minor D_4 of degree 4 of D. By Laplace's expansion, D_4 is a rational function of the minors of degree 2. Applying (1) for $n=4$, $i_1=i_2=\sigma$, we conclude the same for $a^2{}_{t\sigma}D_4$. Hence $a^2{}_{t\sigma}=R$, where R is a rational function of the minors of degree 2. Now the minor

$$\begin{vmatrix} a_{t\sigma} & a_{tl} \\ a_{k\sigma} & a_{kl} \end{vmatrix} = a_{t\sigma}\,a_{kl} - R_lR'_k\,a^2{}_{t\sigma} = R(a_{kl}/a_{t\sigma} - R_lR'_k).$$

Hence $a_{kl} \div a_{t\sigma}$ is a rational function of the minors of degree 2. *Every element a is a rational function of* $\sqrt{R}$ *and the minors.*

6. In conclusion I establish a general theorem on determinants which includes (4) as one special case, and Studnicka's theorem (cited in §1) as another very special case. Let $r_1, r_2, \ldots, r_{m-1}$ be any distinct integers of the set $1, \ldots, n$; likewise for $i_{11}, i_{12}, \ldots, i_{1\,m-1}, j_1$. Let r be any integers $\gtreqless n$. In

$$M_{rj_1} \equiv \begin{vmatrix} a_{ri_{11}} & a_{ri_{12}} \cdots\cdots a_{ri_{1m-1}} & a_{rj_1} \\ a_{r_1 i_{11}} & a_{r_1 i_{12}} \cdots\cdots a_{r_1 i_{1m-1}} & a_{r_1 j_1} \\ \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{11}} & a_{r_{m-1} i_{12}} \cdots a_{r_{m-1} i_{1m-1}} & a_{r_{m-1} j_1} \end{vmatrix}$$

call $A_{ri_{11}}, \ldots, A_{rj_1}$ the co-factors of the elements of the first row. Call C_i the ith column in $D \equiv |a_{ij}|$. Multiply $C_{i_{11}}$ by $A_{ri_{11}}$, $C_{i_{12}}$ by $A_{ri_{12}}, \ldots, C_{i_{1m-1}}$ by $A_{ri_{1m-1}}$, and add the products to $C_{j_1} \times A_{rj_1}$. Hence $A_{rj_1} D$ equals a determinant whose elements outside column C_{j_1} are the same as in D, while the element lying in the j_1th column and rth row is M_{rj_1}, whence those in the rows $r_1, \ldots, r_{m-1}$ are zero. Next, let $i_{21}, i_{22}, \ldots, i_{2m-1}, j_2$ be any distinct integers other than j_1 of the set $1, \ldots, n$. In

$$\mathrm{M}_{rj_2} \equiv \begin{vmatrix} a_{ri_{21}} & a_{ri_{22}} & \cdots\cdots a_{ri_{2m-1}} & a_{rj_2} \\ a_{r_1 i_{21}} & a_{r_1 i_{22}} & \cdots\cdots a_{r_1 i_{2m-1}} & a_{r_1 j_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{21}} & a_{r_{m-1} i_{22}} & \cdots\cdots a_{r_{m-1} i_{2m-1}} & a_{r_{m-1} j_2} \end{vmatrix}$$

call $A_{ri_{21}}, \ldots, A_{rj_2}$ the co-factors of the elements of the first row. Add $C_{i_{21}} A_{ri_{21}}, \ldots, C_{i_{2m-1}} A_{ri_{2m-1}}$ to $C_{j_2} A_{rj_2}$. Hence $A_{rj_1} A_{rj_2} D$ equals a determinant whose elements outside columns C_{j_1} C_{j_2} are the same as in D, while the elements in columns j_1 and j_2 and rth row are M_{rj_1} and M_{rj_2}, respectively. For the third step of the argument, $i_{31}, \ldots, i_{3m-1}, j_3$ are any distinct integers other than j_1, j_2 of the set $1, \ldots, n$. For the $(n-m+1)$th step, $i_{n-m+1}, \ldots, i_{n-m+1\,m-1}$, j_{n-m+1} are any distinct integers other than $j_1, j_2, \ldots, j_{n-m}$ of the set $1, \ldots, n$. After this final step, $A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m+1}} D$ equals a determinant whose elements outside columns $C_{j_1}, \ldots, C_{j_{n-m+1}}$ are the same as in D, while the elements in these columns and the rth row are $M_{rj_1}, \ldots, M_{rj_{n-m+1}}$, respectively. In particular, the latter elements are obviously zero for $r = r_1, \ldots, r_{m-1}$. By Laplace's development the determinant is, apart from sign, $A_{rj_{n-m+1}} \times$ co-factor. The sign is determined by the fact that our co-factor is gotten by deleting the rows $r_1, \ldots, r_{m-1}$ and the columns $i_{n-m+1\,1}, \ldots, i_{n-m+1\,m-1}$. Thus

$$A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m}} D = \pm \; | M_{rj_s} | \tag{6}$$

$(s=1, \ldots, n-m+1;\ r=1, \ldots, n;\ r \neq r_1, \ldots, r_{m-1})$, where the diagonal term of the determinant A_{rj_s} is $a_{r_1 i_{s1}} a_{r_2 i_{s2}}, \ldots, a_{r_{m-1} i_{sm-1}}$.

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